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DUALITY AND THE MAXIMAL FLOW CAPACITY
OF A GENERAL NETWORK

by

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13. ABSTRACT

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II

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III

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1. INTRODUCTION

Ford and Fulkerson (1956) observed that an easy way to find the value of the maximal possible flow through an undirected source-sink planar network* is to construct the dual graph, assign the capacities of the intersected primal arcs as lengths of the undirected arcs in the dual, and then find the shortest route through the dual. The length of this shortest route will be equal to the value of the minimum cut set of the primal network and hence is equal to the value of the maximal flow because of the min-cut max-flow theorem.

Extension of this idea to directed networks began with de Ghellinck (1961). He developed a convention for constructing a dual of a directed source-sink planar network but was not concerned with arc lengths. In a paper written in 1969, Doulliez and Rao (1971) provide the dual construction and the arc length conventions for directed capacitated flow networks when zero lower bounds exist on arc flows. This convention was apparently the result of conversations with de Ghellinck.

Because of a need for dual graphs of directed network in studies of network interdiction models, the author (1970) independently developed the same convention as Doulliez and Rao but went one step further and provided a convention for handling problems where the lower bound on arc flow could take any negative value or any positive value not exceeding the upper bound capacity. This paper also observed that for

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*Ford and Fulkerson (1956) defined source-sink planar networks as planar networks which remain planar after an arc connecting the source and sink is added to the network.

flow networks having positive lower bounds on arc flows, an infeasible primal problem is indicated by a cycle of negative length in the dual network. In a later paper (1971) the author proved that a cycle of negative length was a necessary and sufficient condition for infeasibility. In addition, a proof was provided that the value of the minimal feasible flow could be obtained by changing the sign on the arc length of the dual and finding the longest route from the destination back to the origin.

During 1970, M. F. Sakarovitch was also working on these same problems. His research (1970) was motivated by some conversations with de Ghellinck. He also proved that a cycle of negative length was a necessary and sufficient condition for infeasibility. In addition, he proved that an optimal flow in a primal arc of a maximal flow network is equal to the difference between the dual shortest distances from the origin to the two nodes at opposite ends of the dual arcs intersecting the primal arc. This important property will play a major role in the algorithm to be presented below for non-planar directed networks.

All of the work just summarized is restricted to those networks which are source-sink planar. The main purpose of this paper is to extend these ideas to general flow networks.

In the sections to follow, we will first review the "state of the art" for source-sink planar directed network flow problems. We will then turn to the problem of non-planar networks. A pseudo dual will be defined and the special form of the dual shortest route problem will be stated. The details of the algorithm for solving this problem

will then be presented followed by an example, and the proofs which validate the algorithm. The final section presents a brief discussion of the usefulness of the algorithm.

2. MAXIMAL FLOW IN A PLANAR NETWORK

The original flow network will be called the primal network. A mesh of a planar primal network is any region surrounded by nodes and arcs but containing neither in the plane on which the network is constructed. The region of the plane completely surrounding the primal network will be called the external mesh.

The construction of the dual of a source-sink planar directed network consists of the following steps (McMasters (1970)):

1. Connect an artificial arc between the sink and source of the primal and position it below the network such that it crosses no other arcs. The resulting network will be referred to as the modified primal network.

2. Place a node in each mesh of the modified primal including the external mesh. Let the origin of the dual be the node in the mesh involving the artificial arc and the destination be the node in the external mesh.

3. For each arc in the primal (except the artificial arc) construct two oppositely directed arcs that intersects it and join with nodes in the meshes adjacent to it.

4. Assign the value of the upper bound capacity of the primal arc as the length of the intersecting dual arc having the same direction that the primal arc would have if it were rotated 90 degrees counterclockwise. Assign to the oppositely directed dual arc a length equal to the negative of the lower bound capacity of the primal arc.

The convention described in step 4 is illustrated in Figure 1 where the arc (i,j) is the primal arc having its flow X_{ij} constrained between two values L_{ij} and M_{ij} such that $L_{ij} \leq X_{ij} \leq M_{ij}$. The adjacent dual nodes are u and v. Arc (u,v) is assigned a length equal to M_{ij} while arc (v,u) is assigned a length equal to $-L_{ij}$. We assume $M_{ij} \geq 0$ but place no restriction on L_{ij} except that it cannot exceed M_{ij}.

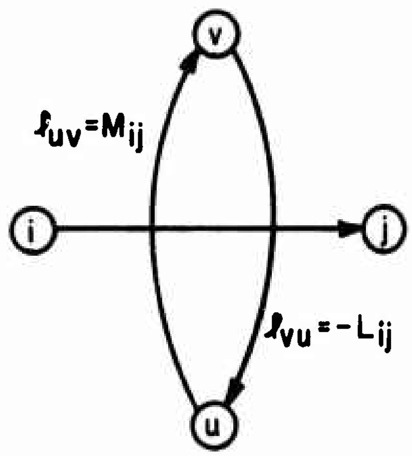


Figure 1.

Figure 2a is an example of a flow network having node 1 as its source and node 4 as its sink. The numbers on each are represented L_{ij}, M_{ij}. Figure 2b shows the initial phase of the construction of the dual. The dual nodes are designated as A, B, C and D. Each dashed arc corresponds to the location of each pair of dual arcs. Figure 2c is the completed dual with its associated arc lengths.

Ford and Fulkerson (1962) defined the generalized value of a primal cut set when lower bounds are positive by equation (1) where

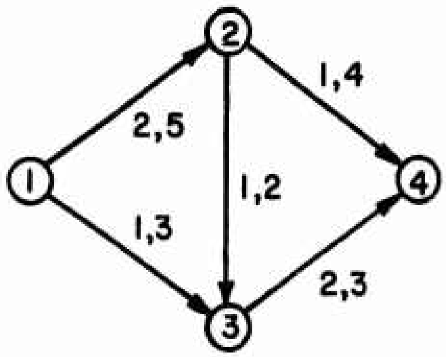


Figure 2a.

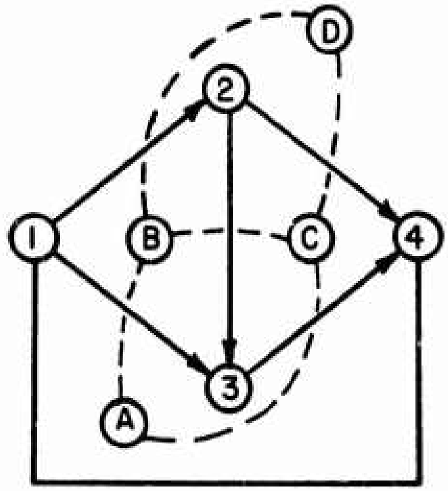


Figure 2b.

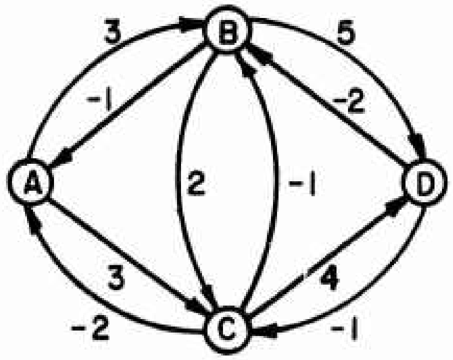


Figure 2c.

the set S contains the primal source node and the set $\bar{S}$ contains the primal sink node.

$$V(S,\bar{S}) = \sum_{\substack{i\varepsilon S \\ j\varepsilon \bar{S}}} M_{ij} - \sum_{\substack{i\varepsilon \bar{S} \\ j\varepsilon S}} L_{ij}. \qquad (1)$$

They then extended the min-cut max-flow theorem to include this definition of the value of a cut set; namely, if a feasible flow can be found for a network with nonnegative lower bounds on arc flows

then the maximal possible flow through the network is equal to the minimal generalized value of all possible cut sets of the $(S,\bar{S})$ type. Later Fulkerson (1962) dropped the restriction that the lower bound be nonnegative.

The relationship between the convention of Figure 1 and the generalized value of a primal cut set is described by the following theorem:

Theorem 1: Any dual chain without cycles directed from the dual source to the dual sink has a length equal to the generalized value of the primal cut set intersected by the arcs of the chain.

Proof: Each path without circuits through the dual of a source-sink planar undirected graph intersects a cut set of the primal network such that the primal source and sink are separated by the cut and there is a separate path corresponding to each cut (Ford and Fulkerson (1956), Whitney (1933)). This is still true if the primal and dual arcs are replaced by directed arcs. Further, because the dual graph under the Figure 1 convention has two oppositely directed arcs between every pair of nodes in adjacent primal meshes it is also true that a separate chain exists directed from the dual source to the dual sink corresponding to each primal cut.

Suppose we select an arbitrary cut set of the primal network such that the source is in a set S and the sink is in set $\bar{S}$. Any such cut set can be illustrated as shown in Figure 3 for planar graphs.

Notice that this set contains both arcs directed from S to $\bar{S}$ and from $\bar{S}$ to S. Let the chain directed from the dual source to the dual sink which intersects the cut set arcs be $a - b - c - d - e - f$.

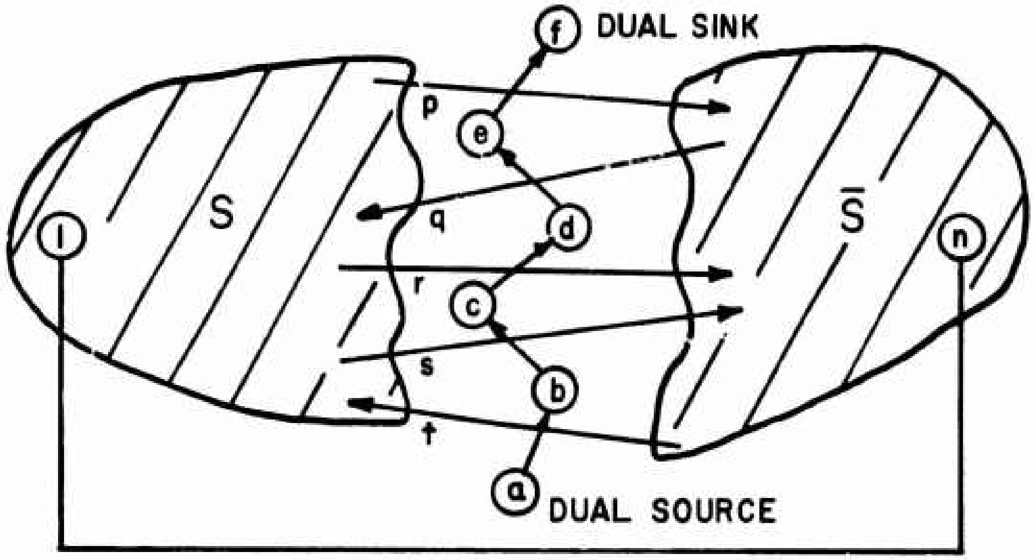


Figure 3.

Using the convention of Figure 1 we assign a length of $-L_t$ to dual arc (a,b) which crosses primal arc t, a length of M_s to arc (b,c) which crosses arc s, a length of M_r to arc (c,d), a length of $-L_q$ to arc (d,e), and a length of M_p to arc (e,f). The chain directed from a to f would therefore have a length of

$$M_s + M_r + M_p - L_t - L_q$$

which we see immediately corresponds to the form of the right-hand side of (1) for this arbitrary cut set (the arcs p, r and s have $i \epsilon S$ and $j \epsilon \bar{S}$ while arcs q and t have $i \epsilon \bar{S}$ and $j \epsilon S$). Because our cut was arbitrary we realize that if we analyze any cut set of the $(S,\bar{S})$ type in this manner we will get the length for its dual chain

being described by this general form. Therefore, each chain without cycles from the dual source to the dual sink will have a length equal to the generalized value of the primal cut set it intersects.

As a consequence of Theorem 1 and the generalization of the min-cut max-flow theorem we get the following corollary:

Corollary 1: The value of maximal flow in a feasible directed source-sink planar network is equal to the length of the shortest route through the dual network.

Recalling Figure 2a we know that the value of the primal cut set consisting of arcs (1,2), (2,3), and (3,4) is

$$M_{12} + M_{34} - L_{23} = 5 + 3 - 1 = 7. \qquad (2)$$

The route through the dual network which corresponds to this cut set is the chain [(A,C),(C,B),(B,D)] which has a length given exactly by equation (2). This chain happens to be a minimum route of Figure 2c so that the maximum possible flow through the primal network is, in fact, 7 units. As we will show below, the fact that we found a shortest route of finite length means that the primal problem is feasible.

The extension of the min-cut max-flow theorem involving the use of the generalized value of a cut set as given by (1) is contingent on the existence of a feasible flow. Feasibility requires that a set of arc flows can be found which satisfy the flow conservation equations at each node as well as the upper and lower bounds on individual arc flows. When the lower bound is nonpositive then we know that a flow

problem will always be feasible since $X_{ij} = 0$ is a feasible solution under the $M_{ij} \geq 0$ assumption. If, however, the lower bounds on some arcs are positive then we cannot claim such problems will be always feasible. An example of an infeasible problem can be obtained by changing the lower bound on arc (1,3) in Figure 2a to a value of 3.

The answer to the feasibility question is provided by Theorem 2.

Theorem 2: A necessary and sufficient condition for an infeasible flow problem associated with a directed source-sink planar network is that a cycle of negative length exists in the dual network.

The proof of Theorem 2 depends upon the circulation theorem of Hoffman (1960) which is addressed to networks having a sourceless-sinkless structure. Such a network is easily created from any flow network by connecting a directed arc from the sink to the source and specifying the appropriate upper and lower bounds on its arc flow.

The circulation theorem states that a network flow problem will be feasible if and only if

$$\sum_{\substack{i \in Y \\ j \in \bar{Y}}} M_{ij} \geq \sum_{\substack{i \in \bar{Y} \\ j \in Y}} L_{ij} \tag{3}$$

for all cut sets $(Y,\bar{Y})$ of the network after it has been converted to the sourceless-sinkless form.

Suppose we define the value of the cut set $(Y,\bar{Y})$ as

$$V(Y,\bar{Y}) = \sum_{\substack{i\epsilon Y \\ j\epsilon \bar{Y}}} M_{ij} - \sum_{\substack{i\epsilon \bar{Y} \\ j\epsilon Y}} L_{ij}. \tag{4}$$

If we do this then the feasibility condition given by (3) is equivalent to $V(Y,\bar{Y}) \geq 0$ and infeasibility corresponds to some $V(Y,\bar{Y}) < 0$.

An interesting correspondence between dual cycles and $V(Y,\bar{Y})$ is described by Theorem 3. We will prove this theorem before proving Theorem 2 since it will be useful in the proof of the latter.

Theorem 3: There is a separate dual cycle intersecting arcs of each primal cut set $(Y,\bar{Y})$. *This cycle has a length equal to* $V(Y,\bar{Y})$.

Proof: Any circuit in the dual corresponds to a cut set $(Y,\bar{Y})$ of the primal (Whitney (1933)). Because of our convention of having two oppositely directed dual arcs intersect each primal arc we know that two cycles exist (oppositely directed) in the dual corresponding to each cut set of the $(Y,\bar{Y})$ type in the primal. Suppose the set $\bar{Y}$ is circumscribed by these dual cycles. The convention of Figure 1 will then result in the cycle directed clockwise having a length corresponding to $V(Y,\bar{Y})$. If Y is the set circumscribed by the cycles then the counterclockwise cycle has a length $V(Y,\bar{Y})$.

Figure 4 is an example of an arbitrary cut set of the $(Y,\bar{Y})$ type with its associated dual cycles. Primal arcs e, f, h and j are directed from Y to $\bar{Y}$; arcs g, i, k and ℓ are directed from $\bar{Y}$ to Y. The convention of Figure 1 will result in the cycle p-q-r-s-t-u-v-w-p having a length of

$$M_g + M_i + M_k + M - L_e - L_f - L_h - L_j$$

which is the right hand side of (4).

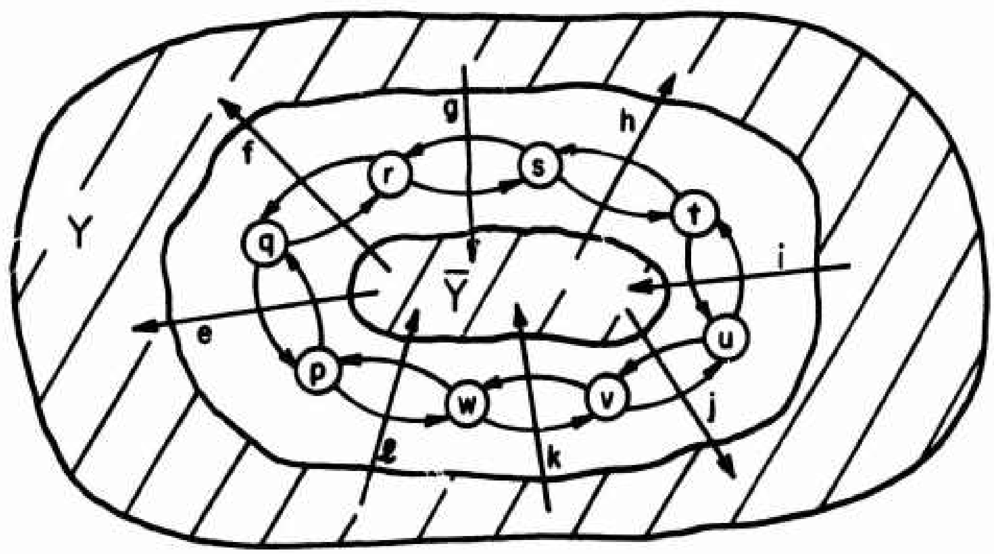


Figure 4.

We will now prove Theorem 2. Suppose that the primal problem is infeasible. From the circulation theorem we know that there will be some cut set $(Y,\bar{Y})$ of the sourceless-sinkless primal which has

$$\sum_{\substack{i\in Y \\ j\in \bar{Y}}} M_{ij} < \sum_{\substack{i\in \bar{Y} \\ j\in Y}} L_{ij}. \tag{5}$$

We can rewrite (5) in the following form:

$$\sum_{\substack{i\in Y \\ j\in \bar{Y}}} M_{ij} - \sum_{\substack{i\in \bar{Y} \\ j\in Y}} L_{ij} < 0. \tag{6}$$

From (4) we know that the left-hand side of (6) is the value of the cut set $(Y,\bar{Y})$. It is also the length of a cycle in the dual because of Theorem 3. Therefore, an infeasible primal problem results in a cycle of negative length in the dual.

To show sufficiency, we consider a cycle of negative length in the dual. Let its length be represented by D. Because of Theorem 3,

$$D = \sum_{\substack{i \in Y \\ j \in \bar{Y}}} M_{ij} - \sum_{\substack{i \in \bar{Y} \\ j \in Y}} L_{ij} = V(Y,\bar{Y}),$$

and D being negative means that the value of some cut set $(Y,\bar{Y})$ in the primal is negative. For a cut set $(Y,\bar{Y})$ to have a negative value (5) must be satisfied and the primal will be infeasible.

3. MINIMAL FLOW IN A PLANAR NETWORK

It is sometimes useful to also know the minimal feasible flow for a network having positive lower bounds on arc flows. The following theorem provides an easy way of determining such a flow.

Theorem 4: If the arc lengths of the dual of a feasible primal flow network have their signs reversed then the longest chain directed from the dual destination back to the dual source will have a length equal to the value of the minimum feasible flow through the network.

The proof of this theorem is based on the min-flow equivalent of the max-flow min-cut theorem (Ford and Fulkerson (1962)); that is,

$$\min Q = \max \sum_{\substack{i \varepsilon S \\ j \varepsilon \bar{S}}} L_{ij} - \sum_{\substack{i \varepsilon \bar{S} \\ j \varepsilon S}} M_{ij} , \tag{7}$$

where Q represents the total flow leaving the network.

The combination of our convention of allocating dual arc lengths and the sign reversal just described will cause each chain from the destination back to the origin in the dual network to have a length equal to

$$\sum_{\substack{i \varepsilon S \\ j \varepsilon \bar{S}}} L_{ij} - \sum_{\substack{i \varepsilon \bar{S} \\ j \varepsilon S}} M_{ij} .$$

The proof of this property parallels that of Theorem 1. Therefore, we get the right-hand side of (7) when we find the longest chain back through the dual.

4. OPTIMAL ARC FLOWS IN PLANAR NETWORKS

The final aspect of the source-sink planar network analysis is the determination of the optimal arcs flows of the primal network. These flows are available from the solution to the dual network shortest route or longest route problem depending on whether we want maximal or minimal flow through the network.

Consider the maximal flow problem. Let A be the set of all dual arcs associated with a given primal flow network. Then the dual network shortest route problem can be stated as follows:

$$\text{minimize} \quad \sum\sum_A \ell_{ij} U_{ij},$$

$$\text{subject to} \quad \sum_{j=1}^{N} [U_{ji} - U_{ij}] = \begin{cases} -1 & (i = 1), \\ 0 & (i = 2,3,\ldots,N-1), \\ +1 & (i = N), \end{cases} \tag{8}$$

$$\text{and} \quad U_{ij} \geq 0 \qquad ((i,j)\varepsilon A).$$

If we think of U_{ij} and ℓ_{ij} as representing respectively a flow and a unit flow cost through a dual arc (i,j) then we realize that (8) corresponds to a minimal cost network flow problem for a total flow out of one unit. The optimal solution will yield a chain flow of one unit over the cheapest route from the origin to the destination of the dual network. This route obviously corresponds to the shortest route through the dual network. The dual linear programming problem associated with (8) can be written as

$$\text{maximize} \quad V_N - V_1,$$
$$\text{subject to} \quad V_j - V_i \le \ell_{ij} \qquad ((i,j)\varepsilon A). \tag{9}$$

If we apply any of the current algorithms (see Dreyfus (1969)) for determining the shortest routes from node 1 to every other node we will obtain a number f_i associated with the length of the shortest route from node 1 to node i. If we let $V_i = f_i$ then we will have an optimal solution to (9).

The relationship between the V_i values and the optimal arc flows in the original primal network is described by Theorem 5 (Sakarovitch (1970)).

Theorem 5: Consider a primal arc (α,β) *directed from node* α *to node* β *which has a flow* $X_{\alpha\beta}$. *Let the intersecting dual arcs be* (i,j) *and* (j,i) *with the length of* (i,j) *being* $M_{\alpha\beta}$ *and the length of* (j,i) *being* $-L_{\alpha\beta}$. *If* $X_{\alpha\beta}$ *satisfies*

$$X_{\alpha\beta} = V_j - V_i \tag{10}$$

for every arc (α,β) *then an optimal feasible solution has been found to the primal maximal flow problem.*

Proof: Because $V_j - V_i$ is a feasible solution to (9) we know that

$$L_{\alpha\beta} \le V_j - V_i \le M_{\alpha\beta}$$

under our convention for assigning dual arc lengths. Thus

$$L_{\alpha\beta} \le X_{\alpha\beta} \le M_{\alpha\beta}.$$

and therefore $X_{\alpha\beta}$ is a feasible flow through arc (α,β).

To check for flow conservation at the primal nodes we first introduce a new primal arc (n,1) connecting the sink to the source and set its upper and lower bounds as well as the arc flow from the sink to the source at the value of $V_N - V_1$. We then have a sourceless-sinkless structure and the flow conservation equations can be written as

$$\sum_{\beta} \left[X_{\alpha\beta} - X_{\beta\alpha}\right] = 0 \tag{11}$$

for $\alpha = 1,2,\ldots,n$.

If we next add dual arcs to intersect the primal (n,1) arc we will create a dual network which has a cycle circumscribing every primal node. If we assign arc lengths to these arcs according to our convention, we will be adding two more constraints to (9). However, the previous optimal solution to (9) is also optimal for this modification.

If we sum the $V_j - V_i$ differences over all arcs of any of these dual cycles we will get zero since each V_i appears twice in the summation, once with a plus sign and once with a minus sign. Therefore, $X_{\alpha\beta} = V_j - V_i$ results in the sum of the flows entering and leaving a primal node being equal to zero and (11) is satisfied.

Finally, the optimal solution to (9) will result in $V_j - V_i = \ell_{ij}$ for all arcs on the shortest route between nodes 1 and N of the dual and equation (10) will result in the flows in the arcs

of the primal minimum cut set $(S,\bar{S})$ being $M_{\alpha\beta}$ if $\alpha\epsilon S$ and $\beta\epsilon\bar{S}$ and $L_{\alpha\beta}$ if $\alpha\epsilon\bar{S}$ and $\beta\epsilon S$. No flow augmenting paths will therefore exist between the primal source and sink and hence the existing flows are optimal. This completes the proof of Theorem 5.

The reader will recall from Theorem 4 that we also can get the minimal flow if a problem is feasible by changing the signs on the arc lengths of the dual and finding the longest route from the sink back to the source. The associated longest route linear programming problem can be stated as

$$\text{maximize} \quad \sum\sum_A \ell_{ij} U_{ij},$$

$$\text{subject to} \quad \sum_{j=1}^{N} [U_{ij} - U_{ji}] = \begin{cases} +1 & (i = N), \\ 0 & (i = 2,3,\ldots,N-1), \\ -1 & (i = 1), \end{cases} \tag{12}$$

$$\text{and} \quad U_{ij} \geq 0.$$

The corresponding dual linear programming problem is

$$\text{minimize} \quad W_N - W_1,$$

$$\text{subject to} \quad W_i - W_j \geq \ell_{ij} \qquad ((i,j)\epsilon A). \tag{13}$$

If we determine the longest route from each node to the origin, we will have an optimal feasible solution to (13). The value of $W_N - W_1$ is minimal feasible flow through the network (a negative value means flow from the sink back to the source). The optimal feasible solution to the original primal minimal flow problem can then be obtained from the results of Theorem 6. The proof of Theorem 6 parallels that of Theorem 5.

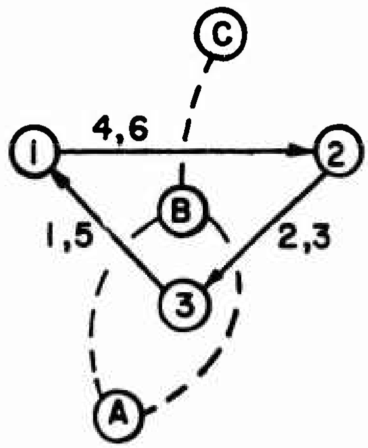
Theorem 6: Consider a primal arc (α,β) *directed from node* α *to node* β *which has a flow* $X_{\alpha\beta}$. *Let the intersecting dual arcs be* (i,j) *and* (j,i) *with the length of* (i,j) *being* $-M_{\alpha\beta}$ *and the length of* (j,i) *being* $L_{\alpha\beta}$. *If* $X_{\alpha\beta}$ *satisfies*

$$X_{\alpha\beta} = W_j - W_i$$

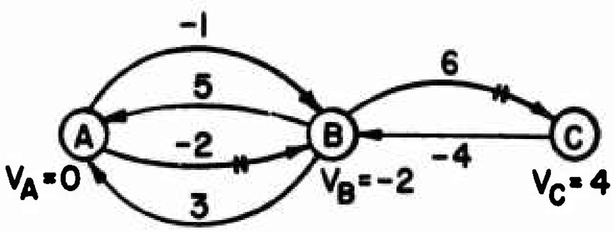
for every arc (α,β) *then an optimal feasible solution has been found to the primal minimal flow problem.*

5. AN EXAMPLE

The results of Theorems 1 through 6 will be illustrated by the following example. Figure 5a is a network having node 1 as its source and node 2 as its sink. The numbers on the arcs represent the lower and upper bounds on arc flow. The dual construction is shown by the nodes A, B and C and the dashed arcs connecting them. The resulting dual network for the maximal flow problem is shown in Figure 5b. The maximal flow through the primal is 4 corresponding to $V_C - V_A$. The dual shortest route is the chain of arcs in Figure 5b having the hash marks. The primal minimum cut set is therefore $\{(1,2),(2,3)\}$.



(a)



(b)

Figure 5.

The flow through arc (1,2) of the primal is 6 as given by the difference $V_C - V_B$ since the dual arcs (B,C) intersect it. The flow through arcs (2,3) and (3,1) are both 2 corresponding to the difference $V_A - V_B$.

If L_{31} is changed to a value of 4 then the top (A,B) arc of Figure 5b has a length of -4. A cycle of negative length will exist consisting of the top (A,B) arc and the bottom (B,A) arc. The problem is infeasible according to Theorem 2. Inspection of Figure 5a under this change shows that an infeasibility condition has been created by arc (3,1) having a lower bound exceeding the capacity of arc (2,3).

Figure 6 is the dual network for the minimal flow problem.

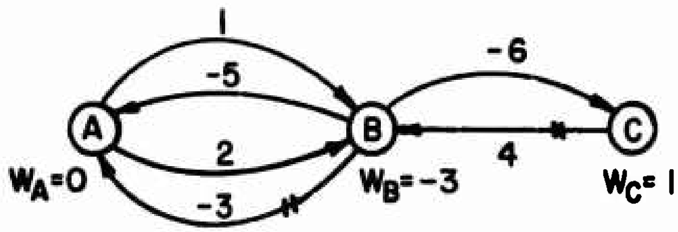


Figure 6.

The minimum feasible flow from the source to the sink of the primal is $W_C - W_A = 1$. The arc flows creating this are $X_{12} = W_C - W_B = 4$ and $X_{23} = X_{31} = W_A - W_B = 3$.

6. MAXIMAL FLOW IN NON-PLANAR NETWORKS

We would like to make use of the ideas developed in the preceding section for all networks and not just those which are source-sink planar. The purpose of this section is to present a way of approaching such problems.

Suppose we construct a two-dimensional representation of a non-planar network such that all arcs are straight lines. If we do this, then no two arcs will intersect more than once at points which are not nodes. If we designated these extra arc intersection points as pseudo nodes we will have created a network which is source-sink planar and we can construct its dual network using the procedure and convention of section 2.

Unfortunately, we have no guarantee that the shortest route through this dual network will give the maximal feasible flow and that the associated optimal feasible arc flows can be determined from Theorem 5. Consider Figure 7. Suppose arc (α,β) and (γ,δ) intersect at pseudo node ϕ.

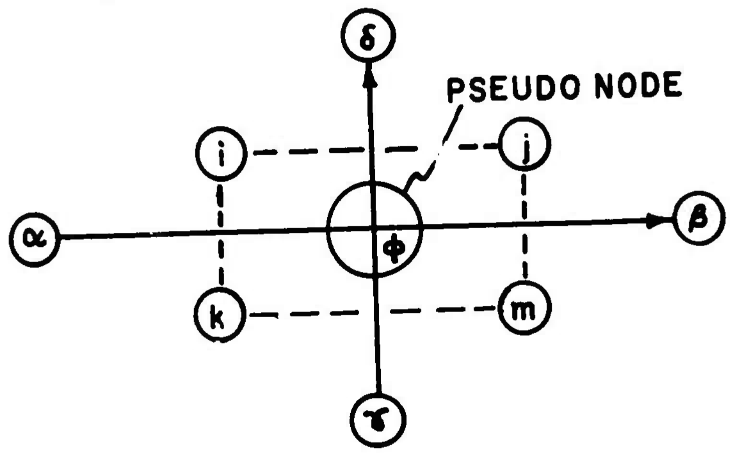


Figure 7.

If we hope to use the results of Theorem 5 to obtain the optimal feasible flows in these arcs, we must require that the values V_i, V_j, V_k, and V_m in (9) for the associated dual nodes satisfy

$$\begin{aligned} V_i - V_k &= V_j - V_m, \\ V_j - V_i &= V_m - V_k. \end{aligned} \tag{14}$$

Condition (14) is needed because an optimal feasible solution to the original primal maximal flow problem must have

$$X_{\alpha\phi} = X_{\phi\beta},$$

$$X_{\gamma\phi} = X_{\phi\delta},$$

since arcs (α,β) and (γ,δ) do not intersect in the original primal network. Therefore, for Theorem 5 to be useful, condition (14) must be added to the constraints of (9).

We note in passing that (14) can be consolidated into only one equation which is

$$V_j + V_k - V_i - V_m = 0. \tag{15}$$

Let the node group surrounding a primal pseudo node ϕ be represented by $(i,j,k,m)_\phi$. Let Φ be the set of all pseudo nodes in a given primal network. The incorporation of (15) into (9) then results in the following problem:

$$\text{minimize} \quad V_N - V_1$$

$$\text{subject to} \quad V_j - V_i \leq \ell_{ij} \qquad ((i,j)\varepsilon A), \qquad (16)$$

$$V_j + V_k - V_i - V_m = 0 \qquad ((i,j,k,m)_\phi; \;\; \phi\varepsilon\Phi)$$

Obviously, the optimal solution to (16) cannot always be determined using an algorithm for finding the shortest route from node 1 to each node i. The following algorithm is, therefore, suggested as a way of solving (16).

Algorithm:

1. Compute the shortest route lengths from the dual origin to all nodes. If, in the process, a cycle of negative length is detected, terminate the algorithm. The primal flow problem is infeasible. If no cycle of negative length exists then compute the value of $V_j + V_k - V_i - V_m$ for $(i,j,k,m)_\phi$ associated with each pseudo node in Φ and go to step 2.
2. Examine the set of $V_j + V_k - V_i - V_m$ values obtained in step 1.
   (a) If $V_j + V_k - V_i - V_m = 0$ for all pseudo nodes in Φ then terminate. The optimal solution to the primal network problem can be obtained by applying Theorem 5 to the planar representation of that network.
   (b) If $V_j + V_k - V_i - V_m \neq 0$ for any set $(i,j,k,m)_\phi$ then compute a set of numbers Y_i $(i = 1,2,\ldots,N)$ where $Y_N = V_N$ and

$$Y_i = \max_j \, (Y_j - \ell_{ij})$$

   for $i = 1,2,\ldots,N-1$. Go to step 3.

3. For each set $(i,j,k,m)_\phi$ for which $V_j + V_k - V_i - V_m \neq 0$ compute the following numbers:

$$\ell_1(\phi) = \min\{\ell_{ij}, V_j - Y_i, V_m - Y_k\},$$

$$\ell_2(\phi) = \max\{-\ell_{ji}, Y_j - V_i, Y_m - V_k\}.$$

If $\ell_1(\phi) < \ell_2(\phi)$ for one or more sets then go to step 4; otherwise go to step 5.

4. Determine that set $(i,j,k,m)_{\hat{\phi}}$ corresponding to

$$\ell_2(\hat{\phi}) - \ell_1(\hat{\phi}) = \max\{\ell_2(\phi) - \ell_1(\phi) \mid \ell_1(\phi) < \ell_2(\phi)\}$$

and change ℓ_{ij}, ℓ_{ji}, ℓ_{km} and ℓ_{mk} of the arcs associated with that set to

$$\ell'_{ij} = \ell'_{km} = \ell_1(\hat{\phi}),$$

$$\ell'_{ji} = \ell'_{mk} = -\ell_1(\hat{\phi}).$$

Return to step 1.

5. Select any set $(i,j,k,m)_\phi$ for which $\ell_1(\phi)$ and $\ell_2(\phi)$ values have been computed and change ℓ_{ij}, ℓ_{ji}, ℓ_{km} and ℓ_{mk} of the arcs associated with that set to

$$\ell'_{ij} = \ell'_{km} = \ell_1(\phi)$$

$$\ell'_{ji} = \ell'_{mk} = -\ell_1(\phi).$$

Return to step 1.

7. AN EXAMPLE

To illustrate the algorithm we will solve for max Q in the example shown in Figure 8. This network, while being planar, is not source-sink planar. Figure 8 is, in fact, the two dimensional representation. We see that all arcs are straight lines and one pseudo node has been created. The numbers on the arcs are the M_{ij} values; $L_{ij} = 0$ for all arcs.

The dual network is shown in Figure 9. The V_i and Y_i values are shown next to their respective nodes. We see immediately that $V_f + V_e - V_d - V_g \neq 0$ so the Y_i values are needed. Next we obtain $\ell_1(\phi) = 5$ and $\ell_2(\phi) = 8$. Therefore, we change ℓ_{df} and ℓ_{eg} to 5, ℓ_{fd} and ℓ_{ge} to -5 and recompute the shortest route.

The new set of V_i values are shown in Figure 10. We now have $V_f + V_e - V_d - V_g = 0$ so we are done. The maximal possible flow through the primal network is 7 as indicated by the difference $V_h - V_a$. The optimal solution to the maximal flow problem associated with the primal network is provided by Figure 11. The numbers on the arcs represent X_{ij}/M_{ij}. The minimum cut set is $\{(4,5),(4,2),(1,2),(3,2),(5,3)\}$.

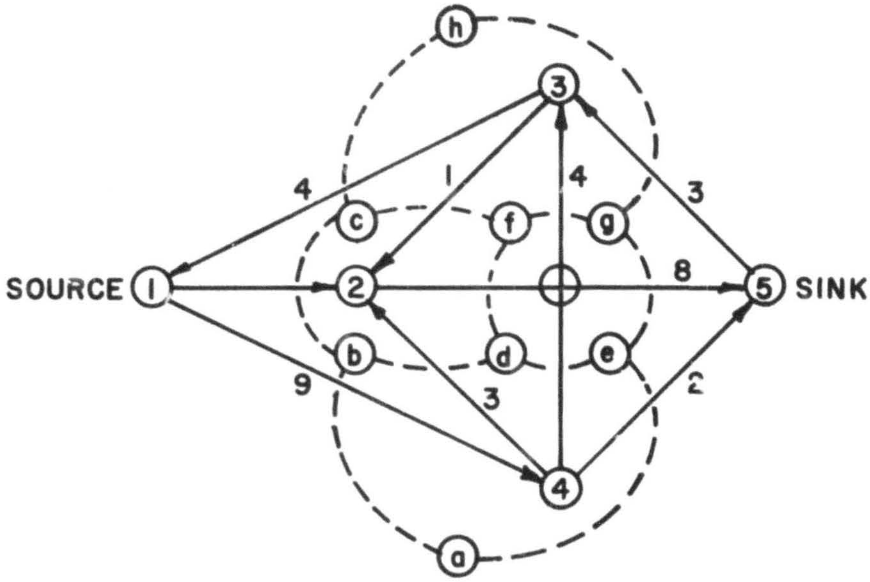


Figure 8.

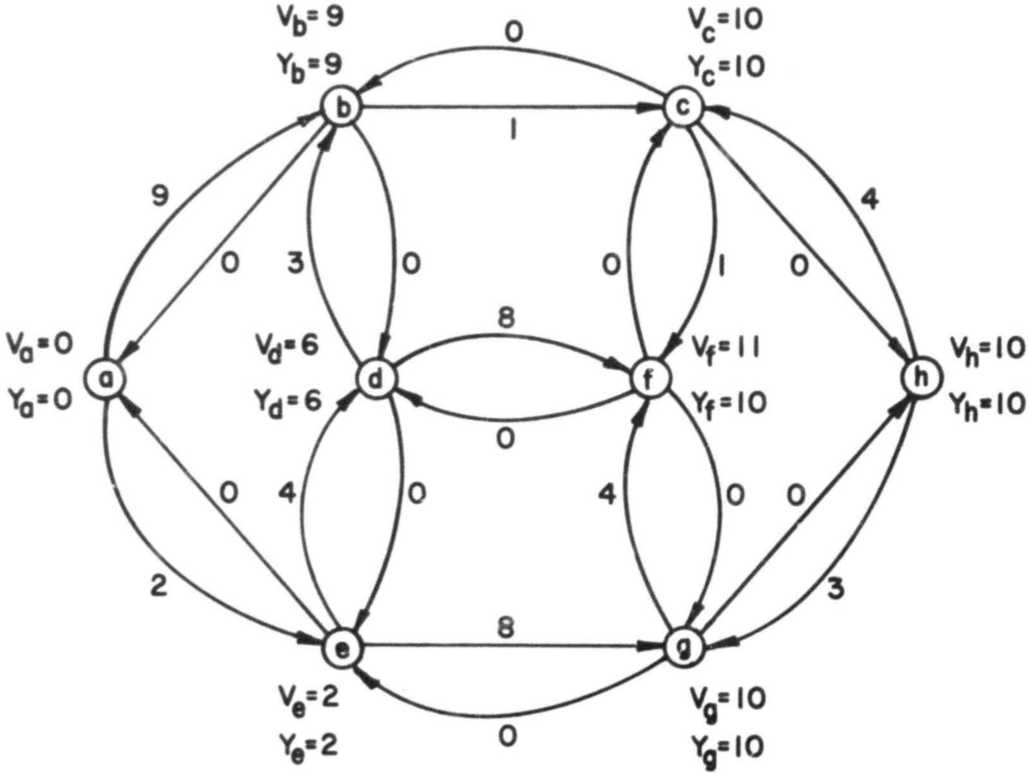


Figure 9.

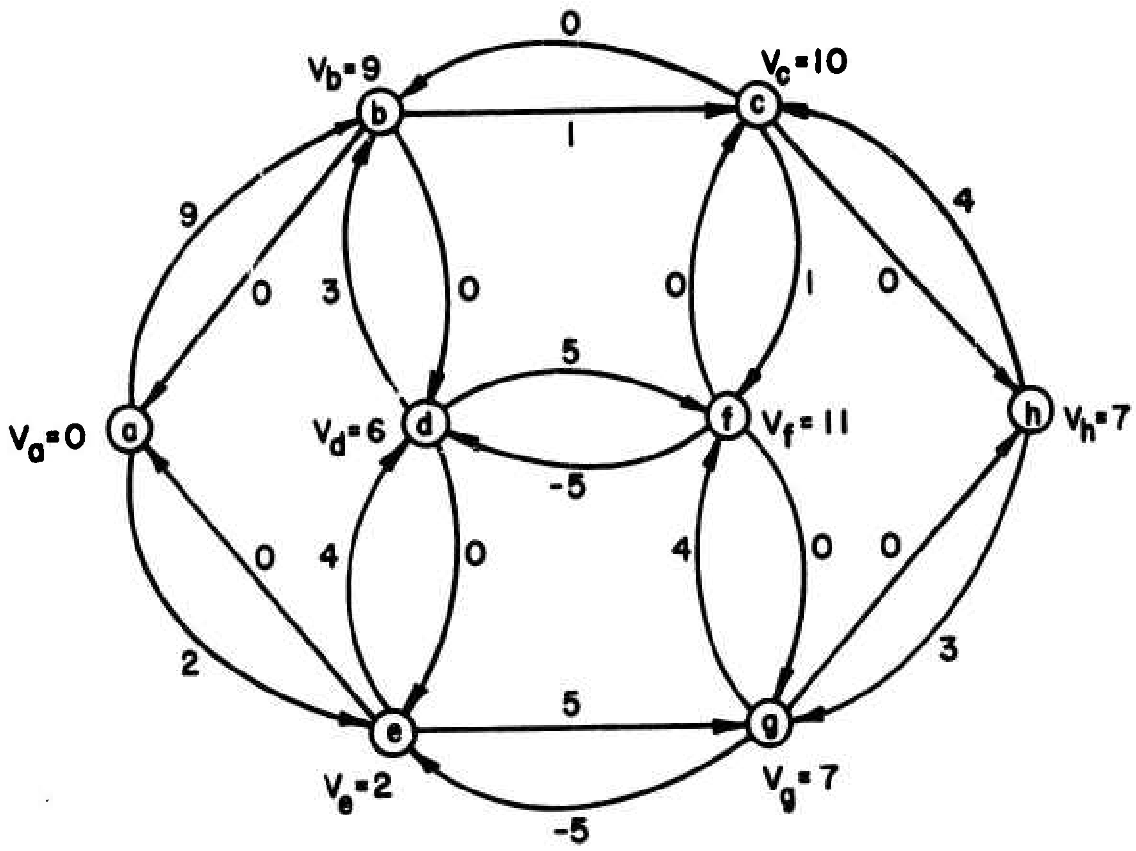


Figure 10.

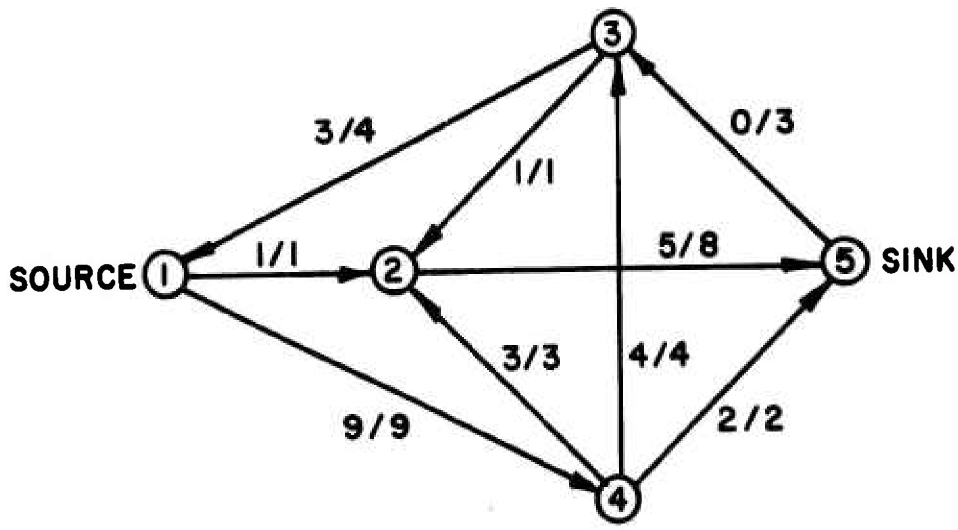


Figure 11.

8. PROOF OF THE ALGORITHM

When $V_j + V_k - V_i - V_m \neq 0$ for some set $(i,j,k,m)_\phi$ we must seek new values, V_i', V_j', V_k', V_m', which will satisfy the constraints of (16). In an attempt to avoid reducing the value of $V_N - V_1$ and, therefore, the value of $\max Q$ we first compute the Y_i values. The differences $Y_N - Y_i$ $(i = 1,2,\ldots,N-1)$ are the lengths of the shortest routes from each of the nodes to node N. Now

$$V_N - V_i \leq Y_N - Y_i, \tag{17}$$

since $V_N - V_i$ will be less than or equal to the length of any chain directed from node i to node N (equality holds only if node i is on the shortest route from node 1 to node N). Because $Y_N = V_N$, (17) reduces to

$$Y_i \leq V_i.$$

Therefore, if we select V_i' satisfying the constraints of (16) such that

$$Y_i \leq V_i' \leq V_i \tag{18}$$

then we are assured that $V_N - V_1$ will not change.

Combining (18) with the inequality constraints of (16) results in the following ranges for $V_j' - V_i'$ and $V_m' - V_k'$:

$$\begin{aligned} \max\{-\ell_{ji}, Y_j - V_i\} &\leq V_j' - V_i' \leq \min\{\ell_{ij}, V_j - Y_i\}, \\ \max\{-\ell_{mk}, Y_m - V_k\} &\leq V_m' - V_k' \leq \min\{\ell_{km}, V_m - Y_k\}. \end{aligned} \tag{19}$$

Let us assume the ranges in (19) have some values in common. We next combine the inequalities of (19) with the equality constraints of (16). The result is (20). Notice that ℓ_{mk} and ℓ_{km} have been omitted. This is due to the fact that $\ell_{ij} = \ell_{m\ell}$ and $\ell_{ji} = \ell_{mk}$ for every $(i,j,k,m)_{\phi}$ as a consequence of the dual construction.

$$\max\{-\ell_{ji}, Y_j - V_i, Y_m - V_k\} \leq V'_j - V'_i = V'_m - V'_k \leq \min\{\ell_{ij}, V_j - Y_i, V_m - Y_k\}. \quad (20)$$

We realize that (20) reduces to the following form:

$$\ell_2(\phi) \leq V'_j - V'_i = V'_m - V'_k \leq \ell_1(\phi),$$

when we use the definitions of $\ell_1(\phi)$ and $\ell_2(\phi)$ from step 2 of the algorithm.

Step 4 of the algorithm handles the problems where the ranges of (19) have values in common for all ϕ which violate (15). It changes the arc lengths of arcs (i,j) and (k,m) to $\ell_1(\phi)$ and the arc lengths of (j,i) and (m,k) to $-\ell_1(\phi)$. This forces the conditions

$$V_m - V_k = V_j - V_i = \ell_1(\phi),$$

$$V_i - V_k = V_j - V_m,$$

while satisfying (18). Note that one node of each pair, (i,j) and (k,m), will have $V'_i = V_i$ after the new computations. Finally, $V'_N - V'_1 = V_N - V_1$ so that the value of maximal flow has not been affected by the new computations.

If the ranges of (19) have no values in common for some set $(i,j,k,m)_\phi$ then $\ell_1(\phi) < \ell_2(\phi)$ and the current $V_N - V_1$ value represents an infeasible value for max Q. Because V_N must be reduced by the amount

$$\ell_2(\hat{\phi}) - \ell_1(\hat{\phi}) = \max_{\phi}\{\ell_2(\phi) - \ell_1(\phi) \mid \ell_1(\phi) < \ell_2(\phi)\}$$

before all sets $(i,j,k,\ell)_\phi$ can meet the constraints of (16), step 3 adjusts the appropriate arc lengths of the set or sets corresponding to $\hat{\phi}$ to $\ell_1(\hat{\phi})$ and $-\ell_1(\hat{\phi})$ to provide this reduction in V_N immediately. No further use of step 3 is necessary after this reduction. It is possible, however, that adjustments of the step 4 variety will still be needed before an optimal solution is reached.

Changing the arc lengths of the arcs (i,j), (k,m) and (j,i), (m,k) to $\ell_1(\phi)$ and $-\ell_1(\phi)$ values or to $\ell_1(\hat{\phi})$ and $-\ell_1(\hat{\phi})$ values, respectively, results in the optimal solution to the primal having a partial flow of $\ell_1(\phi)$ or $\ell_1(\hat{\phi})$ in the intersected primal arc (γ,δ).

The arcs whose lengths have been changed either remain as members of the tree of shortest routes from node 1 to all nodes (the early tree) or become members in the next round of V_i calculations. Only when all are members of an early tree will the condition $V_j - V_i = V_m - V_k$ be satisfied if it was not satisfied initially.

The forcing of the reverse arc of a pair to a value of $-\ell_1(\phi)$ or $-\ell_1(\hat{\phi})$ does not create a primal infeasibility (that is, a cycle of negative length in the dual) for the following reasons. Suppose arc (i,j) was originally on the early tree. The condition $L_{\gamma\delta} \leq M_{\gamma\delta}$

means that the minimum possible value of ℓ_{ji} will be $-M_{\gamma\delta}$. This will occur only if $L_{\gamma\delta} = M_{\gamma\delta}$ in the primal problem. If arc (i,j) was originally in the early tree then the shortest route from node i to node j is over arc (i,j). Any other route in the dual network will have a length at least as great as $\ell_{ij} \equiv M_{\gamma\delta}$. Thus, even specifying $\ell_{ji} = -M_{\gamma\delta}$ does not create a negative cycle with any of the routes. The worst that happens is that a cycle of zero length exists consisting of the arcs (i,j) and (j,i). If arc (i,j) is now shortened to a value of $\ell_i(\phi)$ or $\ell_1(\hat{\phi})$ then it remains on the tree and ℓ_{ji} will have a value $-\ell_1(\phi)$ or $-\ell_1(\hat{\phi})$ which is larger than $-M_{\gamma\delta}$ so no negative cycle is created.

Next, suppose arc (i,j) was not originally in the early tree. There must, therefore, be some other route between nodes i and j which is shorter (it might be arc (j,i)). Reducing the length of (i,j) to $\ell_1(\phi)$ or $\ell_1(\hat{\phi})$ causes it to become an alternate member of the early tree. Changing the length of arc (j,i) to a length of $-\ell_1(\phi)$ or $-\ell_1(\hat{\phi})$ creates a cycle of zero length with arc (i,j). If the original shortest route between nodes i and j was not the arc (j,i) but rather some other route directed from node i to node j then its length is at least $\ell_1(\phi)$ or $\ell_1(\phi)$ or arc (i,j) would not be the alternative shortest route after the new V_i computations. Therefore, at worst a cycle of zero length would be created by setting ℓ_{ji} at a value of $-\ell_1(\phi)$ or $-\ell_1(\hat{\phi})$. If, instead, the original shortest route was some route directed from node j to node i then a change in ℓ_{ji} has no effect since it is not a member of a cycle containing the original shortest route. Reduction of ℓ_{ij} to the value of $\ell_1(\phi)$ or $\ell_1(\hat{\phi})$ will at worst create a cycle of zero length when combined with this route.

9. COMMENTS ABOUT THE ALGORITHM

The intent of the algorithm of section 6 is to provide a means of solving dual networks for primal network optimal flows without the worry of whether the network is source-sink planar. All that is required is that the primal network be drawn such that no arc intersects any other arc more than once. While the algorithm is designed only to solve maximal flow problems, the necessary modifications for solving the minimal flow problem are minor and obvious.

This algorithm may not be as efficient an approach as the maximal flow algorithm of Ford and Fulkerson even if no pseudo nodes are needed because the dual graph must be constructed. The existence of pseudo nodes could reduce efficiency further since several iterations of the algorithm may be required before an optimal flow is found or the primal network is found to be infeasible. Careful construction of the two-dimensional representation of the primal can, however, reduce pseudo nodes to a minimum.

The algorithm is quite useful and efficient for sensitivity analyses and parametric studies of network flow as a function of primal arc capacities and for problems involving capacity expansion and reduction (see, for example, Doulliez and Rao (1971) and McMasters and Mustin (1970)). In such problems, the dual network needs to be constructed only once. The changes in primal arc capacities then appear as arc length changes in the dual and a new shortest route can be quickly determined when necessary for each change.

REFERENCES

[1] Doulliez, P. J. and M. R. Rao, "Capacity of a Network with Increasing Demands and Arcs Subject to Failure," Operations Research, Vol. 19, No. 4, July-August 1971.

[2] Dreyfus, S. E., "An Appraisal of Some Shortest Path Algorithms," Operations Research, Vol. 17, No. 3, May-June 1969.

[3] Ford, L. R. and D. R. Fulkerson, "Maximal Flow Through a Network," Canad. Jour. Math., Vol. 8, No. 3, 1956.

[4] Ford, L. R. and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, New Jersey, 1962.

[5] Fulkerson, D. R., "Flow Networks and Combinatorial Operations Research," RAND Memorandum RM-3378-PR, October 1962.

[6] de Ghellinck, G., "Aspects de la Notion de Dualité en Théorie des Graphes," Cahiers de Recherche Operationelle, Vol. 3, No. 2, 1961.

[7] Hoffman, A. J., "Some Recent Applications of the Theory of Linear Inequalities to Extremal Combinatorial Analysis," Proc. Symposia on Applied Math., Vol. 10, 1960.

[8] McMasters, A. W., "Appraising Feasibility and Maximal Flow Capacity of a Network," Research Rpt. NPS55MG70101A, Naval Postgraduate School, Monterey, October 1970.

[9] McMasters, A. W., "Appraising Feasibility and Maximal Flow Capacity of a Network," Paper FP8.20, 39th National ORSA Meeting, Dallas, May 1971.

[10] McMasters, A. W. and T. M. Mustin, "Optimal Interdiction of a Supply Network," NRLQ 17, 261-267 (1970).

[11] Sakarovitch, M. F., "A Note on Duality in Network Flows," Report ORC 70-36, Opns. Res. Ctr., Univ. of Calif., Berkeley, November 1970.

[12] Whitney, H., "Planar Graphs," Fundamenta Mathematicae 21, 73-84 (1933).